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ON THE USE OF QUATERNIONS IN SIMULATION OF RIGID-BODY MOTION

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FOREWORD

The work covered by this report was done in the System Dynamics Branch, Aeronautical Research Laboratory, under Project 7060, "Flight Dynamics Research and Analysis Facility". Mr. Paul W. Nosker is Project Engineer. This study is part of a continuing program to determine optimum methods of simulation and analysis of the dynamics of air weapon systems. The general subject of quaternions as applied to coordinate conversions has been under investigation for approximately two years, though the bulk of the work reported here was accomplished during the last six months of 1957.

The author wishes to express his appreciation to Mr. Robert T. Harnett and others of the Analog Computation Branch of the Aeronautical Research Laboratory for assistance in the analog simulation portion of the study.

Note: This is a re-creation of the original report, done in 2005-6 using a scanner, OCR software and a word processor, and with the following differences of detail from the original:

- Only sections I to V and Appendices A and B are included
- The general appearance (e.g. font types and sizes) is similar to the original report but not identical;
- The page breaks occur at different places;
- Spelling and punctuation have been corrected in a few places;
- Text that I judge to be incorrect or incomplete has been amended (shown in brackets like [these]);
- Some corrections to the mathematics have been made (e.g. 2 sign changes in equation (12)).
- Some footnotes have been added for clarification and to indicate the mathematical corrections made. These are indicated by my initials [OW]
- Most footnotes are numbered instead of being indicated by asterisks;

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SECTION I

INTRODUCTION

The problem of motion of a rigid body and the associated one of coordinate conversion are very old ones in the field of classical dynamics. Significant results, dating from the time of Euler (1776) through the introduction and application of matrix methods by Cayley and Klein and others in the last half of the nineteenth century, brought the matter to such a satisfactory state that no significantly new methods or approaches have been found necessary. The development of modern computing machinery makes necessary a re-examination of the various methods from the standpoint of their utility in computational devices. It is not necessarily true that methods which have proven their convenience in the largely analytical manipulations of classical mechanics should prove to be best adapted for numerical or analog computation. Quaternions fell into disuse among physicists about the turn of the present century because matrix and vector methods had proved more useful in the types of investigation then being conducted. The purpose of the present paper is to show that the quaternion approach to coordinate transformation does offer real advantages in the analog simulation of rigid body motion. In recent times Deschamps and Sudduth[1] have suggested an application for digital computation, and Backus[2] has proposed them for analog simulations, but in general quaternions are little known among those engaged in simulation of aircraft motions.

The coordinate conversion problem in aircraft and missile simulation is different at least in emphasis from that of classical dynamics. It might be well to state the problem which is of interest and to which the methods explained later will be applied. A missile or aircraft may be considered as a moving coordinate system. Various vectors must be transformed into this coordinate system or out of it into some inertial systems. Integrating the equations of motion of the airframe can be made to yield the three components of the coordinate system's angular velocity vector. From the X, Y and Z components (P, Q, R) of this vector in the moving system, it is desired to keep track of the orientation of the coordinate system in such a way that vectors may be transformed in either direction. This means an integration of angular rate to determine angular position.

Fundamental to this procedure is a consideration of how the orientation of the coordinate system is to be specified. During the history of the subject, various methods of doing this have been put forward. All the most useful ones fall into three categories: Euler angles, quaternions, and direction cosines. Of these, the first and last are probably the most familiar to modern readers. In the Euler angle method, the orientation is expressed as the result of three rotations

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[1] Deschamps, G. A. and W. B. Sudduth, Federal Telecommunications Laboratories, Nutley, New Jersey, Case 26-10707, November 1955

[2] Backus, George, Rigid Body Equations - Euler Parameters, Technical Note 6, Advisory Board on Simulation, University of Chicago, November 1951

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about each of three axes, the rotations being made in a specific sequence. The physical interpretation of a quaternion is a rotation through some angle about some specific fixed axis. The nine direction cosines are simply the cosines of the angles between each of the axes in the moving system with each of the axes of the fixed system. Principal attention here will be given to the quaternion, or four-parameter system. It was first introduced by Euler in 1776, as a result of spherical trigonometry considerations. The elegant quaternion formulation was invented by Hamilton in 1843 as a new kind of algebraic formalism. A matrix formulation was devised by Klein for use in gyroscopic problems and, in this formulation, is usually known as the Cayley-Klein parameters. Each of these three different approaches to the four-parameter system has its own advantages. It has been decided to present at least an outline of all three here. There are two reasons for this: first, there are some propositions which are more easily shown by one development: second, it seems probable that when the reader is offered a choice of method, he will reach an understanding sooner if he can select the method most nearly consonant with his own background.

It will become apparent that this subject presents something of an expositional problem. In order to reach the desired ends, it has been decided to assume that the reader has a knowledge of matrix methods, especially as applied to coordinate conversion in three-dimensional space. As a compromise, a brief introduction to the subject is given in Appendix A, though a more satisfactory treatment is given by Goldstein[*]. In this report the term “quaternion” has been used to represent the four-parameter method in general. In other cases, it is necessary to use the word to distinguish Hamilton’s development from the others. It is hoped that confusion may be kept to a minimum.

There are many different techniques used in present-day aircraft simulations to solve the coordinate conversion problem. The technique is usually adapted to the special requirements of the problem at hand. If most of the rotation takes place about one axis, or if only the gravity vector is to be handled, or if the airframe’s rotation is otherwise restricted, valuable simplifications may be effected in the analog equipment required to represent the conversion. It is not the present purpose, however, to investigate all these possibilities. Consideration will be given only to the most general and unrestricted case: that of several complete revolutions about any or all axes. This immediately excludes the Euler angles because of the singular point. The advantages of Euler angles are such, and their popularity is so pervasive, however, as to warrant keeping them in mind. Accordingly, Appendix B gives a brief outline of the Euler Angle system most commonly used in aircraft work, and at appropriate points, comparisons will be made of them with quaternions and direction cosines. In making such comparisons, that form of Euler angle instrumentation whose capabilities most nearly equal those of the quaternion scheme will be assumed. This form has been discussed at some length by Howe[3] and his figures and results will be used for comparison. In Howe’s method, the extent and direction of rotation is

[*] Goldstein, Herbert, Classical Mechanics, Addison Wesley Press, Cambridge, Mass., 1950.

[3] Howe, R. M. and E. G. Gilbert, A New Resolving Method for Analog Computers, WADC Technical Note 55-468, January 1956.

unrestricted except for the inevitable singular orientation, and he shows that even this leads to less practical difficulties than one might expect.

It is valuable to keep the Euler angles in mind, but the quaternion method must really stand or fall on its comparison with direction cosines. It has in common with direction cosines the capability of handling completely unrestricted rotations. Accordingly, considerable attention has been devoted to the direction cosine method in this report. Both a theoretical error analysis and a simulation program were done for the cosines in order to provide the most complete possible basis of comparison. They have been done before, but it is difficult to compare results obtained by different investigators on different computing equipment. An attempt was made here to keep the conditions as nearly comparable as possible. Of all the material contained herein, no originality is claimed except for the quaternion error analysis and simulation. Even here, no new techniques were used, with the possible exception of the method of handling multiplier errors. It was felt necessary, however, to include the remaining material in order to introduce and place in context this probably unfamiliar subject.

SECTION II

THE EULER PARAMETERS

The earliest formulation of the four-parameter system was given by Euler in 1776, though the oldest treatment generally available today is probably that of Whittaker[4]. It is an essentially geometrical development, but will not be presented as such here. The principal results may be demonstrated with much less labor by use of matrices.

Central to the development of these parameters, and indeed to the four-parameter methods in general, is the proposition known as Euler's theorem, which may be stated as follows: any real rotation may be expressed as a rotation through some angle, about some fixed axis. In other words, regardless of what the rotation history of a body is, once it reaches some orientation, that orientation may be specified in terms of a rotation through some angle (which can be determined) about some fixed axis.

The truth of this proposition is not intuitively obvious, but in any case, it must be shown. Consider a transformation matrix (A). No restrictions are put on (A) other than those which exist for all orthogonal transformation matrices (see Appendix A). Another way of stating Euler's theorem is to say that for every matrix[5] (A) there exists some vector $\overline{R}$ whose components are the same before and after application of (A); in other words there must be some $\overline{R}$ such that

$$(A)\overline{R} = \overline{R} \tag{1}$$

for any (A). If the components of $\overline{R}$ are designated X, Y and Z, the elements of (A) by a_{mn}, then Equation (l) may be written

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \tag{2}$$

If this matrix equation is expanded in components, a set of linear homogeneous equations results:

$$\begin{aligned} (a_{11}-1)X + a_{12}Y + a_{13}Z &= 0 \\ a_{21}X + (a_{22}-1)Y + a_{23}Z &= 0 \\ a_{31}X + a_{32}Y + (a_{33}-1)Z &= 0 \end{aligned} \tag{3}$$

A necessary and sufficient condition for existence of a non-trivial solution is that the determinant of coefficients be zero. Therefore, it is necessary to show that

$$\begin{vmatrix} a_{11}-1 & a_{12} & a_{13} \\ a_{21} & a_{22}-1 & a_{23} \\ a_{31} & a_{32} & a_{33}-1 \end{vmatrix} = 0\,. \tag{4}$$

[4] Whittaker, E. T. Analytical Dynamics, Fourth Edition, Dover Publications, N. Y., 1944

[5] Orthogonal transformation matrix [OW]

This may easily be done making use of the properties of an orthogonal transformation matrix developed in Appendix A. If the above equation is expanded,

$$(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{21}a_{32}a_{13}-a_{31}a_{13}a_{22}-a_{21}a_{12}a_{33}-a_{32}a_{23}a_{11}-1)$$
$$+(a_{11}-a_{22}a_{33}+a_{23}a_{32})+(a_{22}-a_{11}a_{33}+a_{13}a_{31})+(a_{33}-a_{11}a_{22}+a_{21}a_{12}) = 0 \qquad (5)$$

The first term vanishes in consequence of the fact that the determinant of the transformation matrix must equal unity (Equation (156)), and the last three terms vanish from the orthogonality conditions[6] of Equation (162).

Thus, it is proved that Equation (4) is an identity for any orthogonal (A) and that there exists some vector R which is unchanged by the transformation. This proves Euler's theorem.

Since it has been shown that it is possible to express any rotation as a single rotation about some axis, it is possible to make use of the equivalent rotation to specify orientation. Consider two coordinate systems X Y Z and X' Y' Z'. The X Y Z system is assumed to be fixed in inertial space, and X' Y' Z' is moving in some arbitrary manner, though both coordinate systems have the same origin. Assume that initially the two systems are coincident. Then the X' Y' Z' system is rotated through an angle μ about an axis which makes angles α, β, γ with the X, Y, Z, axes respectively. It will be noted that this axis of rotation makes the same angles α, β, γ with the X', Y', Z' axes also. It is now necessary to express the transformation matrix in terms of the quantities μ, α, β and γ.

In order to do this, use is made of an additional coordinate system, X_r Y_r Z_r, which is fixed in the X Y Z system. The X_r axis lies along the axis of rotation, and the Yr axis is restricted to the XY plane. This would give rise to difficulty if the Z axis is the axis of rotation, but in that case, the Y_r axis could be confined to the XZ plane or the YZ plane, and the final result would be unaltered. At any rate, with the choice indicated, the Y_r axis is always perpendicular to the Z axis. Now the rotation through the angle μ is a rotation through μ about the X axis, so 'the rotation is a very simple one in the $X_rY_rZ_r$ system. Accordingly, the rotation of the X'Y'Z' system through the angle μ may be viewed as the result of three rotations: (1) rotation of the X'Y'Z' system into coincidence with the $X_rY_rZ_r$ system; (2) rotation through the angle μ about the X_r axis; (3) the reverse of (1) to restore the original separation of the X'Y'Z' and $X_rY_rZ_r$ systems. The matrix for each of these transformations will be developed, and then the three may be multiplied together to express the total transformation.

First, the transformation into the $X_rY_rZ_r$ system will be considered. α, β and γ are the angles between the new X_r axis and the fixed X, Y and Z axes.

Thus, it is seen from Equation (125) that a_{11}, a_{12}, and a_{13} are immediately fixed. One other cosine may be established. Recall that the Y_r axis is perpendicular to the Z axis. This means that $a_{23} = 0$. Thus the matrix of the first rotation is partially established.

[6] 1, 5 and 9 [OW]

$$(A) = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{6}$$

Applying the orthogonality conditions[7], it is possible to deduce that the other elements are

$$(A) = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \mp\cos\beta\,\csc\gamma & \pm\cos\alpha\,\csc\gamma & 0 \\ \mp\cos\alpha\,\cot\gamma & \mp\cos\beta\,\cot\gamma & \pm\sin\gamma \end{pmatrix} \tag{7}$$

The ambiguities in sign may be resolved by making use of the requirement that the matrix above must reduce to the identity matrix when α becomes zero**.** The result is

$$(A) = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ -\cos\beta\,\csc\gamma & \cos\alpha\,\csc\gamma & 0 \\ -\cos\alpha\,\cot\gamma & -\cos\beta\,\cot\gamma & \sin\gamma \end{pmatrix} \tag{8}$$

The second rotation, through the angle MU, about the Xr axis is simply

$$(R) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{pmatrix} \tag{9}$$

The last of the three rotations is the inverse of (A) or $(A)^{-1}$. Thus, the general transformation is the result of all three, called (B). It is given by

$$(B) = (A)^{-1}(R)A) \tag{10}$$

This is a similarity transformation, and, among other things, the spur (sum of the diagonal elements) of a matrix is invariant under a similarity transformation, i.e.,

$$b_{11} + b_{22} + b_{33} = 1 + 2\cos\mu \tag{11}$$

so the angle of rotation may be obtained directly from the diagonal elements of the transformation matrix. Carrying out the operations of Equation (10) gives[8]

$$(B) = \begin{pmatrix} 1 - 2\sin^2\frac{\mu}{2}\sin^2\alpha & \begin{matrix} 2(\sin^2\frac{\mu}{2}\cos\alpha\,\cos\beta \\ +\sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\gamma) \end{matrix} & \begin{matrix} 2(\cos\alpha\,\cos\gamma\,\sin^2\frac{\mu}{2} \\ -\sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\beta \end{matrix} \\ \begin{matrix} 2(\sin^2\frac{\mu}{2}\cos\alpha\,\cos\beta \\ -\sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\gamma) \end{matrix} & 1 - 2\sin^2\frac{\mu}{2}\sin^2\beta & \begin{matrix} 2(\sin^2\frac{\mu}{2}\cos\beta\,\cos\gamma \\ +\sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\alpha) \end{matrix} \\ \begin{matrix} 2(\cos\alpha\,\cos\gamma\,\sin^2\frac{\mu}{2} \\ +\sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\beta) \end{matrix} & \begin{matrix} 2(\sin^2\frac{\mu}{2}\cos\beta\,\cos\gamma \\ -\sin\frac{\mu}{2}\cos\frac{\mu}{2}\cos\alpha) \end{matrix} & 1 - 2\sin^2\frac{\mu}{2}\sin^2\gamma \end{pmatrix} \tag{12}$$

If the following substitutions are made,

$$\xi = \cos\alpha\,\sin\tfrac{\mu}{2}, \quad \eta = \cos\beta\,\sin\tfrac{\mu}{2}, \quad \zeta = \cos\gamma\,\sin\tfrac{\mu}{2}, \quad \chi = \cos\tfrac{\mu}{2} \tag{13}$$

[7] 6 of the 12 conditions [OW]

[8] "-" changed to "+" at start of 2nd line of elements (1,2) and (2,3). [OW]

the matrix of (12) becomes

$$B = \begin{pmatrix} \xi^2 - \eta^2 - \zeta^2 + \chi^2 & 2(\xi\eta + \zeta\chi) & 2(\xi\zeta - \eta\chi) \\ 2(\xi\eta - \zeta\chi) & -\xi^2 + \eta^2 - \zeta^2 + \chi^2 & 2(\eta\zeta + \xi\chi) \\ 2(\xi\zeta + \eta\chi) & 2(\eta\zeta - \xi\chi) & -\xi^2 - \eta^2 + \zeta^2 + \chi^2 \end{pmatrix} \tag{14}$$

These four quantities are called the Euler parameters[9]. It may be seen from their definitions that they obey the relationship

$$\xi^2 + \eta^2 + \zeta^2 + \chi^2 = 1 \tag{15}$$

so they are not all independent. Also, none may lie outside the range ±1.

If the quantities μ, α, β and γ are known, it is a simple matter to compute the Euler parameters and/or the transformation matrix by the method given above. If, on the other hand, the transformation matrix is given, it is also possible to solve for the four parameters, though difficulties arise. A consideration of these difficulties will shed further light on the nature of the Euler parameters. To begin with, it should be stated that the quantities μ, α, β and γ cannot be uniquely determined from the transformation matrix. The reason for this is that even though rotation through a certain angle, about a certain axis, will produce a definite unambiguous orientation, the reverse is not true. If the orientation is given, there are four separate ways in which it could have been obtained by rotation about a fixed axis. Possibly an example will help to clarify this. Assume that the rotation being considered is a rotation through an angle of + 30° about the + X axis. There are three other ways to get to the same position: (1) a rotation through -30° about the - X axis; (2) a rotation through -330° about the + X axis; (3) a rotation through +330° about the - X axis. A further illustration of the possibilities is given in the table following.

	χ	ξ	η	ζ
Case 1	$+\cos\frac{\mu}{2}$	$+\cos\alpha\sin\frac{\mu}{2}$	$+\cos\beta\sin\frac{\mu}{2}$	$+\cos\gamma\sin\frac{\mu}{2}$
Case 2	$+\cos\frac{\mu}{2}$	$(-\cos\alpha)(-\sin\frac{\mu}{2})$	$(-\cos\beta)(-\sin\frac{\mu}{2})$	$(-\cos\gamma)(-\sin\frac{\mu}{2})$
Case 3	$-\cos\frac{\mu}{2}$	$+\cos\alpha(-\sin\frac{\mu}{2})$	$+\cos\beta(-\sin\frac{\mu}{2})$	$+\cos\gamma(-\sin\frac{\mu}{2})$
Case 4	$-\cos\frac{\mu}{2}$	$(-\cos\alpha)\sin\frac{\mu}{2}$	$(-\cos\beta)\sin\frac{\mu}{2}$	$(-\cos\gamma)\sin\frac{\mu}{2}$

The first two cases lead to the same Euler parameters, and the last two lead to a different set which are the negative of the first. All four sets lead to the same transformation matrix.

The relationship between Euler parameters and direction cosines may be derived by equating terms in Equation (14)[10]. The result is

$$4\chi^2 = 1 + a_{11} + a_{22} + a_{33}$$

[9] "the Euler symmetric parameters", I believe. [OW]

[10] and using (15)

$$4\xi^2 = 1 + a_{11} - a_{22} - a_{33} \tag{16}$$

$$4\eta^2 = 1 - a_{11} + a_{22} - a_{33}$$

$$4\zeta^2 = 1 - a_{11} - a_{22} + a_{33}$$

These equations determine the Euler parameters except for sign. The sign must be gotten in another way. From comparison of terms in the matrix it is possible to show that

$$a_{31} - a_{13} = 4\chi\eta$$

$$a_{12} - a_{21} = 4\chi\zeta \tag{17}$$

$$a_{23} - a_{32} = 4\chi\xi$$

Thus, if χ is assumed to be always positive, the signs of the others may be deduced from Equations (17) unless $\chi = 0$. This is the special case of a 180° rotation. There is an additional ambiguity here because the direction of the axis of rotation and the direction of the rotation are completely unrelated. Either a positive or a negative rotation about either the positive or negative axis will give the same result. For this special case, another means would have to be devised for defining the signs, but it hardly seems worthwhile to go into it here. It is not expected that this will lead to any practical difficulties.

SECTION III

THE CAYLEY-KLEIN PARAMETERS

In this development of the four-parameter System, it is found that a 2x2 complex matrix may be used to represent a real rotation, rather than a 3x3 real matrix. Consider such a matrix (H),

$$(H) = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} . \qquad (18)$$

The requirement is placed on this matrix that it be unitary, that is to say the product of (H) and its adjoint must yield the unit matrix. The adjoint is the complex conjugate of the transposed matrix. In addition, it is required that the determinant of the matrix (H) have the value +1. The unitary condition allows ±1 for the determinant, so this is an additional requirement. The unitary condition may be written as

$$\begin{pmatrix} h_{11}^* & h_{21}^* \\ h_{12}^* & h_{22}^* \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \qquad (19)$$

Expanding and equating components gives

$$\begin{aligned} h_{11}^* h_{11} + h_{21}^* h_{21} &= 1 \\ h_{11}^* h_{12} + h_{21}^* h_{22} &= 0 \qquad (20) \\ h_{12}^* h_{11} + h_{22}^* h_{21} &= 0 \\ h_{12}^* h_{12} + h_{22}^* h_{22} &= 1 . \end{aligned}$$

The second and third equations are the same, being merely complex conjugates of each other. The first and fourth equations have no imaginary component, whereas the second (or third) has both real and imaginary parts. Therefore, the three independent equations contain four conditions. These, together with the determinant requirement that $h_{11}h_{22} - h_{21}h_{12} = +1$ make it possible to determine certain relationships among the four quantities h_{mn}. It may be shown that $h_{22} = h_{11}^*$ and $h_{21} = h_{12}^*$ so the matrix may be written as

$$(H) = \begin{pmatrix} h_{11} & h_{12} \\ - h_{12}^* & h_{11}^* \end{pmatrix} . \qquad (21)$$

The quantities h_{11}, h_{12}, h_{22} are usually referred to as the Cayley-Klein parameters. It will be noted that they are complex numbers. While it is convenient to use them as such in analytical operations (and this is the purpose for which Klein developed them) a physical computer must treat complex numbers in terms of their real and imaginary parts. Therefore, it is convenient to introduce four other quantities defined as follows:

$$\begin{aligned} h_{11} &= e_1 + ie_2 \qquad (22) \\ h_{12} &= e_3 + ie_4 \end{aligned}$$

where the e's are all real numbers, and i is the square root of -l. Using these definitions, the matrix (H) may be written as

$$(H) = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} . \tag{23}$$

Now consider another complex matrix (P), which has the form

$$(P) = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{24}$$

where **x,** y and z are real numbers. It will be noted that the matrix (P) is equal to its own adjoint, and thus is said to be self-adjoint or Hermitian. Now consider a transformation of (P) of the form

$$(P)' = (H)(P)(H)^+ \tag{25}$$

where $(H)^+$ designates the adjoint of (H). Since (H) is unitary, $(H)^+ = (H)^{-1}$, so equation (25) is

$$(P)' = (H)(P)(H)^{-1} . \tag{26}$$

This is a similarity transformation. It is shown in Appendix A that the determinant of a matrix is invariant under a similarity transformation. It can also be shown that the Hermitian property and the spur are both invariant under a similarity transformation. Therefore, the transformed matrix (P)' must have have the form

$$(P)' = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix} . \tag{27}$$

The fact that the determinant of (P) must equal the determinant of (P)' gives

$$x^2 + u^2 + z^2 = x'^2 + y'^2 + z'^2 \tag{28}$$

If **x,** y and z are viewed as components of a vector, then Equation (28) is the requirement that the length of the vector remain unchanged. Equation (26) may be written

$$\begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix} = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \begin{pmatrix} e_1 - ie_2 & -e_3 - ie_4 \\ e_3 - ie_4 & e_1 + ie_2 \end{pmatrix} . \tag{29}$$

If the operations of Equation (29) are carried out, it is found that

$$\begin{aligned} x' &= (e_1^2 - e_2^2 - e_3^2 + e_4^2)x - 2(e_1e_2 + e_3e_4)y + 2(e_2e_4 - e_1e_3)z \\ y' &= 2(e_3e_4 - e_1e_2)x + (e_1^2 - e_2^2 + e_3^2 - e_4^2)y + 2(e_2e_3 + e_4e_1)z \\ z' &= 2(e_1e_3 + e_2e_4)x + 2(e_2e_3 - e_1e_4)y + (e_1^2 + e_2^2 - e_3^2 - e_4^2)z . \end{aligned} \tag{30}$$

These equations represent a linear transformation between the components of **x,** y and z, and the components of x' y' and z'. The matrix for this transformation is

$$(A) = \begin{pmatrix} e_1^2 - e_2^2 - e_3^2 + e_4^2 & 2(e_1e_2 + e_3e_4) & 2(e_2e_4 - e_1e_3) \\ 2(e_3e_4 - e_1e_2) & e_2^2 - e_2^2 + e_3^2 - e_4^2 & 2(e_2e_3 + e_4e_1) \\ 2(e_1e_3 + e_2e_4) & 2(e_2e_3 - e_1e_4) & e_1^2 + e_2^2 - e_3^2 - e_4^2 \end{pmatrix} . \tag{31}$$

It may be shown directly that this matrix satisfies the orthogonality conditions, but it is proved also from Equation (28). Equation (31) shows that the nine direction cosines may be expressed in terms of the four e's. If Equations (22) are substituted into Equations (20) it is found that

$$e_1^2 + e_2^2 + e_3^2 + e_4^2 = 1 \tag{32}$$

and therefore, only three of the e's are independent. The identity of these four quantities with the Euler parameters is obvious. Comparison of Equations (3l) and (14) gives

$$e_1 = \chi,\ e_2 = \zeta,\ e_3 = \eta,\ e_4 = \xi\ . \tag{33}$$

An equivalence has been indicated between the real (3x3) matrix (A) and the complex (2x2) matrix (H). It may be shown that this correspondence goes further. Consider the real transformation

$$\bar{r}' = (B)\,\bar{r} \tag{34}$$

and let the associated unitary complex matrix be $(H)_1$, so that

$$(P)' = (H)_1(P)'(H)_1^+ \tag{35}$$

Now consider a second transformation (A) with associated $(H)_2$.

$$\bar{r}'' = (B)\,\bar{r}'$$

$$(P)'' = (H)_2(P)'(H)_2^+ \tag{36}$$

Substituting (34) and (35) into (36) gives

$$\bar{r}'' = (C)\,\bar{r}'$$

$$(P)'' = (H)_2(H)_1(P)(H)_1^+(H)_2^+ \tag{37}$$

Therefore, if (A)(B) = (C) and $(H)_2(H)_1 = (H)_3$, the above equations become

$$\bar{r}'' = (C)\,\bar{r}'$$

$$(P)'' = (H)_3(P)(H)_3^+ \tag{38}$$

showing that multiplication of two real 3x3 matrices corresponds to multiplication of the two associated 2x2 complex matrices in the same order. Two types of quantities which correspond in this manner are said to be isomorphic.

It is also possible to view this process of two successive rotations in terms of the e's themselves. Consider one rotation defined by e_1 e_2, e_3 and e_4. After this, another rotation is performed which is described by e_1', e_2', e_3' and e_4'. There is some set of e's called e_1'', e_2'', e_3'', e_4'' which describes the final orientation after the two rotations. This combined set may be found by multiplying the (H) matrices of the two rotations in the correct sequence. The equation is

$$(H)'' = \begin{pmatrix} e_1'' + ie_2'' & e_3'' + ie_4'' \\ -e_3'' + ie_4'' & e_1'' - ie_2'' \end{pmatrix} = \begin{pmatrix} e_1' + ie_2' & e_3' + ie_4' \\ -e_3' + ie_4' & e_1' - ie_2' \end{pmatrix} \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} \tag{39}$$

Expanding this equation and equating components gives

$$\begin{aligned} e_1'' &= e_1'e_1 - e_2'e_2 - e_3'e_3 - e_4'e_4 \\ e_2'' &= e_2e_1' + e_2'e_1 + e_3'e_4 - e_4'e_3 \\ e_3'' &= e_1'e_3 - e_2'e_4 + e_3'e_1 + e_4'\,e_2 \\ e_4'' &= e_2'e_3 + e_1'e_4 + e_4'e_1 - e_3'e_2 \end{aligned} \tag{40}$$

By use of these equations, successive transformations may be handled in terms of the e's directly.

This technique may be used to determine the relationship between the e's and the Euler angles given in Appendix B. The (H) matrix corresponding to each of the Euler angle rotations may be determined, and the three may be multiplied in the correct order to synthesize the complete transformation. Consider first the (H) matrix corresponding to the first Euler angle, given as ψ in Appendix B. From Equation (179) it is seen that the transformation equations are

$$\begin{aligned} x' &= x\cos\psi + y\sin\psi \\ y' &= -x\sin\psi + y\cos\psi \\ z' &= z. \end{aligned} \tag{41}$$

Equating coefficients of these equations with like coefficients in Equations (30) gives the nine relations[11]

$$\begin{array}{lll} \cos\psi = e_1^2-e_2^2-e_3^2+e_4^2, & -\sin\psi = 2(e_3e_4-e_1e_2), & 0 = 2(e_1e_3+e_2e_4), \\ \sin\psi = 2(e_1e_2+e_3e_4), & \cos\psi = e_1^2-e_2^2+e_3^2-e_4^2, & 0 = 2(e_2e_3-e_1e_4), \\ 0 = 2(e_2e_4-e_1e_3), & 0 = 2(e_2e_3+e_4e_1), & 1 = e_1^2+e_2^2-e_3^2-e_4^2. \end{array} \tag{42}$$

These equations cannot all be satisfied unless $e_3 = e_4 = 0$. If this is true, then

$$\cos\psi = e_1^2 - e_2^2, \qquad \sin\psi = 2e_1e_2, \qquad e_1^2+e_2^2 = 1 \tag{43}$$

Solving these equations for e_1 and e_2, gives

$$e_1 = \cos\frac{\psi}{2}, \quad e_2 = \sin\frac{\psi}{2} \tag{44}$$

so the (H) matrix corresponding to the ψ rotation is

$$(H)_\psi = \begin{pmatrix} \cos\frac{\psi}{2} + i\sin\frac{\psi}{2} & 0 \\ 0 & \cos\frac{\psi}{2} - i\sin\frac{\psi}{2} \end{pmatrix} = \begin{pmatrix} e^{i\sin\frac{\psi}{2}} & 0 \\ 0 & e^{-i\sin\frac{\psi}{2}} \end{pmatrix} \tag{45}$$

By an exactly similar process, it may be shown that the other two matrices are[12]

$$(H)_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \qquad (H)_\phi = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix} \tag{46}$$

Therefore, the entire transformation, which is the result of all three rotations, is

$$(H) = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} = (H)_\phi (H)_\theta (H)_\psi \tag{47}$$

Carrying out the indicated multiplications, and equating components gives

$$e_1 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}$$

---

[11] superscripts of e_3 and e_4 corrected from 3 and 4 to 2 in second equation for cos ψ [OW]

[12] subscript in $(H)_\theta$ corrected from $(H)_e$ [OW]

$$e_2 = \sin\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2} - \cos\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2} \tag{48}$$

$$e_3 = \cos\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2} + \sin\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}$$

$$e_4 = \cos\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2} - \sin\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}$$

SECTION IV

QUATERNIONS

The most brilliant formulation of the four-parameter method was rnade by Hamilton in 1843. He developed a new type of entity called a "quaternion". It is composed of four parts,

$$q = S + ia + jb + kc \tag{49}$$

where S, a, b and c are real numbers, and the indices i, j and k are defined by the following rules;

$$\begin{array}{ll} i^2 = -1 & ij = -ji = k \\ j^2 = -1 & jk = -kj = i \\ k^2 = -1 & ki = -ik = j \end{array} \tag{50}$$

The conjugate of the quaternion q is

$$q^* = S - ia - jb - kc. \tag{51}$$

Using the laws for the indices quoted above, it may be easily shown that

$$qq^* = q^*q = S^2 + a^2 + b^2 + \mathbf{c^2} \tag{52}$$

which is called the length or norm of the quaternion. If this norm is unity, then a special form of quaternion results, a versor. It is possible to make use of these to describe a coordinate transformation. The quantity S is called the real or scalar part of the quaternion, and ia + jb + kc is called the imaginary or vector part. Now assume we have a quaternion whose scalar part is zero. We call this a vector of components **X,** Y and Z,

$$V = iX + jY + kZ. \tag{53}$$

Let us examine the operation

$$q^*Vq = V' \tag{54}$$

where q is a versor. So far there is no particular reason to expect that V' will be a vector, but this turns out to be the case. Equation (54) may be written

$$(S - ia - jb - kc)\,(iX + jY + kZ)\,(S + ia + jb + kc) = V'. \tag{55}$$

When this equation is expanded. making use of the rules for indices, the result is[13]

$$\begin{aligned} V' = {} & i\,\{X\,[S^2 + a^2 - b^2 - c^2] + Y\,[2cS + 2ab] + Z\,[2ac - 2Sb]\} \\ & + j\,\{X\,[2ab - 2cS] + Y\,[+S^2 - a^2 + b^2 - c^2] + Z\,[2aS + 2cb]\} \\ & + k\,\{X\,[2Sb + 2ac] + Y\,[2bc - 2Sa] + Z\,[S^2 - a^2 - b^2 + c^2]\}. \end{aligned} \tag{56}$$

This is simply a coordinate transformation whose transformation matrix is

$$\begin{pmatrix} S^2 + a^2 - b^2 - c^2 & 2(cS + ab) & 2(ac - sb) \\ 2(ab - cS) & S^2 - a^2 + b^2 - c^2 & 2(aS + cb) \\ 2(ac + sb) & 2(bc - Sa) & S^2 - a^2 - b^2 + c^2 \end{pmatrix} \tag{57}$$

The correlations with matrices derived in the two preceding sections are evidently

$$e_1 = \chi = S, \qquad e_2 = \zeta = c, \qquad e_3 = \eta = b, \qquad e_4 = \xi = a\,. \tag{58}$$

---

[13] x corrected to X (3 times). s replaced by S in (56) to (58) [OW]

The matter of two successive rotations may be handled quite easily. Assume that first we transform a vector with the versor q_1 ,

$$q_1^*Vq_1 = V' . \tag{59}$$

Next we apply the versor q_2 ,

$$V'' = q_2^*V'q_2 = q_2^*q_1^*Vq_1q_2 . \tag{60}$$

We now define a new vector $q_1q_2 = q_3$, and wish to find the relationship between q_3 and $q_2^*q_1^*$. We define $q_4 = q_2^*q_1^*$. It may be seen that

$$q_2q_2^*q_1^* = q_2q_4 \tag{61}$$

and since q_2 is a versor, $q_2q_2^* = l$. Therefore, Equation (61) reduces to

$$q_1^* = q_2q_4 . \tag{62}$$

Now we apply q_1 on the left,

$$q_1q_1^* = q_1q_2q_4 = 1 = q_3q_4 \tag{63}$$

that q_4 must equal the conjugate of q_3, This means that

$$V'' = q_3^*Vq_3 \tag{64}$$

Now.observe that the equation $q_3 = q_1q_2$ may be written

$$S_3 + ia_3 + jb_3 + kc_3 = (S_1 + ia_1 + jb_1 + kc_1)(S_2 + ia_2 + jb_2 + kc_2). \tag{65}$$

Expanding this equation and equating components gives

$$\begin{aligned} S_3 &= S_1S_2 - a_1a_2 - b_1b_2 - c_1c_2 \\ a_3 &= S_1a_2 + S_2a_1 + b_1c_2 - c_1b_2 \\ b_3 &= S_1b_2 - a_1c_2 + b_1S_2 + c_1a_2 \\ c_3 &= S_1c_2 + a_1b_2 - b_1a_2 + c_1S_2 . \end{aligned} \tag{66}$$

These equations are identical with Equations (40) which were developed in the same connection by use of the Cayley -Klein parameters. Thus, the quaternion method leads to the same result as the preceeding developments.

SECTION V

INFINITESIMAL TRANSFORMATIONS AND RATE OF ROTATION

The preceding sections have dealt with the four-parameter method of specifying the orientation of a coordinate system. As was stated in Section I, however, the prirnary interest is in determining the orientation from the rate of rotation through a process of integration. Accordingly, it is necessary to relate the rates of change of the four parameters to the rates of rotation of the axis system.

It was shown in Section III that an orthogonal transformation may be represented by a complex matrix having certain properties. It is now of interest to investigate this matrix when an infinitesmal rotation is performed. Let us assume that this infinitesmal rotation consists of a rotation through the angle $\Delta\mu$, about a line which makes angles of α, β and γ with the X**,** Y and Z axes respectively. Recall that the matrix (H) may be expressed

$$(H) = \begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} \tag{67}$$

Applying the geometrical interpretation of the e's gives

$$(H) = \begin{pmatrix} \cos\frac{\mu}{2} + i\cos\gamma\sin\frac{\mu}{2} & \cos\beta\sin\frac{\mu}{2} + i\cos\alpha\sin\frac{\mu}{2} \\ -\cos\beta\sin\frac{\mu}{2} + i\cos\alpha\sin\frac{\mu}{2} & \cos\frac{\mu}{2} - i\cos\gamma\sin\frac{\mu}{2} \end{pmatrix} \tag{68}$$

From this, it is possible to see that the infinitesmal rotation may be represented by[14]

$$(H)_\varepsilon = \begin{pmatrix} 1 + i\frac{\Delta\mu}{2}\cos\gamma & \frac{\Delta\mu}{2}\cos\beta + i\frac{\Delta\mu}{2}\cos\alpha \\ -\frac{\mu}{2}\cos\beta + i\frac{\Delta\mu}{2}\cos\alpha & 1 - i\frac{\Delta\mu}{2}\cos\gamma \end{pmatrix} \tag{69}$$

since $\cos\frac{\Delta\mu}{2} \approx 1,\ \sin\frac{\Delta\mu}{2} \approx \frac{\Delta\mu}{2}$.

It is expected that any matrix representing an infinitesmal rotation will differ only slightly from the identity matrix. This is true of the above matrix, and this may be shown more clearly by writing it as follows:

$$(H)_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\Delta\mu}{2}\begin{pmatrix} i\cos\gamma & \cos\beta + i\cos\varepsilon \\ -\cos\beta + i\cos\alpha & -i\cos\gamma \end{pmatrix} = (I) + (\varepsilon)\ . \tag{70}$$

Now assume that this infinitesmal rotation takes place during a small time interval Δt. If (H) is the matrix at the beginning of the interval, and if (H)' is the matrix at the end of the interval, then the time derivative of (H) may be written as

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[14] Missing "=" added [OW]

$$\frac{d}{dt}(H) = \lim_{\Delta t \to 0} \frac{(H)' - (H)}{\Delta t} \tag{71}$$

The final matrix (H)' may also be viewed as the result of two rotations, first (H) and then $(H)_\varepsilon$. In other words, $(H)' = (H)_\varepsilon \, . \, (H)$. Putting this into the above equation gives

$$\frac{d}{dt}(H) = \lim_{\Delta t \to 0} \frac{(\varepsilon)}{\Delta t}(H) \tag{72}$$

Since (H) is not affected by the time increment, the limiting process refers only to the quantity $\frac{(\varepsilon)}{\Delta t}$,

$$\frac{(\varepsilon)}{\Delta t} = \frac{1}{2}\frac{\Delta \mu}{\Delta t}\begin{pmatrix} i\cos\gamma & \cos\beta + i\cos\alpha \\ -\cos\beta + i\cos\alpha & -i\cos\gamma \end{pmatrix} \tag{73}$$

In the limit, the quantity $\frac{\Delta \mu}{\Delta t}$ is simply the scalar magnitude of the angular velocity vector. If P, Q and R are the components of this velocity vector along the **X,** Y and Z axes, then evidently $\frac{d\mu}{dt}\cos\alpha = P$, $\frac{d\mu}{dt}\cos\gamma = R$, $\frac{d\mu}{dt}\cos\beta = Q$, so that

$$\lim_{\Delta t \to 0} \frac{(\varepsilon)}{\Delta t} = \frac{1}{2}\begin{pmatrix} iR & Q + iP \\ -Q + iP & -iR \end{pmatrix} \tag{74}$$

Therefore, from Equation (72),

$$\frac{d}{dt}(H) = \frac{1}{2}\begin{pmatrix} iR & Q + iP \\ -Q + iP & -iR \end{pmatrix}(H) \tag{75}$$

It is also possible to show, by a straightforward limiting process, that the time derivative of a matrix is also a matrix whose elements are the time derivatives of the elements of the original matrix. Therefore[15],

$$\begin{pmatrix} \dot{e}_1 + i\dot{e}_2 & \dot{e}_3 + i\dot{e}_4 \\ -\dot{e}_3 + i\dot{e}_4 & \dot{e}_1 - i\dot{e}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} iR & Q + iP \\ -Q + iP & -iR \end{pmatrix}\begin{pmatrix} e_1 + ie_2 & e_3 + ie_4 \\ -e_3 + ie_4 & e_1 - ie_2 \end{pmatrix} . \tag{76}$$

Expanding and equating like components gives

$$\begin{aligned} 2\dot{e}_1 &= -e_4P - e_3Q - e_2R, \\ 2\dot{e}_2 &= -e_3P + e_4Q + e_1R, \\ 2\dot{e}_3 &= +e_2P + e_1Q - e_4R, \\ 2\dot{e}_4 &= +e_1P - e_2Q + e_3R . \end{aligned} \tag{77}$$

These are the equations which would be used to compute the four pararneters in an actual simulation. Now if we multiply Equation (76) on the right by the adjoint of (H) the result is

[15] From (67) and (75) [OW]

$$\begin{pmatrix} \dot{e}_1 + i\dot{e}_2 & \dot{e}_3 + i\dot{e}_4 \\ -\dot{e}_3 + i\dot{e}_4 & \dot{e}_1 - i\dot{e}_2 \end{pmatrix} \begin{pmatrix} e_1 - ie_2 & -e_3 - ie_4 \\ e_3 - ie_4 & e_1 + ie_2 \end{pmatrix} = +\frac{1}{2} \begin{pmatrix} iR & Q + iP \\ -Q + iP & -iR \end{pmatrix} . \qquad (78)$$

Again expanding and equating components gives

$$\begin{aligned} P &= 2(-e_4\dot{e}_1 - e_3\dot{e}_2 + e_2\dot{e}_3 + e_1\dot{e}_4), \\ Q &= 2(-e_3\dot{e}_1 + e_4\dot{e}_2 + e_1\dot{e}_3 - e_2\dot{e}_4), \qquad (79) \\ R &= 2(-e_2\dot{e}_1 + e_1\dot{e}_2 - e_4\dot{e}_3 + e_3\dot{e}_4) . \end{aligned}$$

Thus, if the four pararneters and their rates of change are known, the angular velocity may be computed.

APPENDIX A

ORTHOGONAL TRANSFORMATIONS

1. The Independent Coordinates of a Rigid Body

Fundamental to the study of rigid body motions is the determination of how many degrees of freedom it has. Putting it another way, the problem is to determine how many numbers one must specify in order to describe the orientation of the body. In order to do this, it will also be necessary to give a more exact definition to the term "rigid".

Assume that a body is composed of a large number of elementary particles. If the distance between the ith particle and the jth particle r_{ij} is constant [for][16] all particles i and j, then the body is said to be rigid. If all the N particles were independent of each other, it would require 3N coordinates to specify them all. (Three cartesian coordinates are required to specify the position of a point.) The particles are not all independent, however. In fact the position of any particle in the body may be specified by the distances to any three non-collinear points in the body.

[16] [OW: original had: “constant of all particles”]

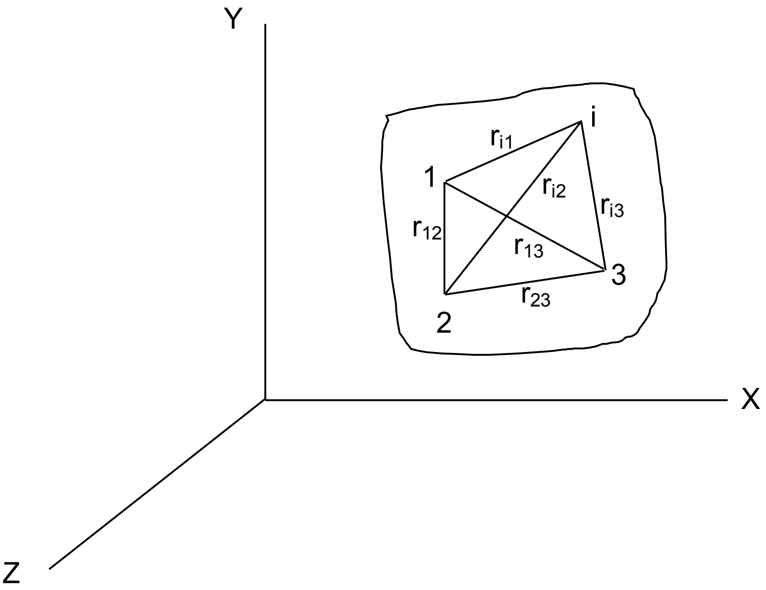

Figure 8

The points 1, 2 and 3 in Figure 8 have been chosen at random, the only condition being that they do not lie along the same [straight] line. By the rigid-body condition that r_{i1}, r_{i2} and r_{i3} are constant, the position of the ith particle is fixed once the positions of the particles [1, 2 and 3 are fixed;][17] it follows that the position of every particle in the body is specified once the three points are specified. In other words, the position of the body is specified by the positions of these three points. Specifying three points would require nine coordinates if all the points were independent. There are three conditions to be fulfilled by these coordinates, however, namely the prescribed values of r_{12}, r_{13} and r_{23}. Thus six coordinates are required to specify the position of the rigid body. Another way of saying this is to say that the rigid body has six degrees of freedom. These are frequently divided into two groups called translational and rotational degrees of freedom. The three coordinates used to specify the orientation of some point in the body (say the point 1 in Figure 8) in the xyz coordinate system, may be called the translational coordinates, while the three coordinates required to specify the relative orientation of the other two points could be called the rotational coordinates. The translational coordinates, then, are associated with the motion of the body as a whole, while the rotational coordinates are associated with the orientation of the body.

2. Orthogonal Transformations

Consider a vector $\vec{r}$ which has components x, y and z in the XYZ coordinate system. If the unit vectors along the X, Y and Z axes are called $\vec{i}$, $\vec{j}$ and $\vec{k}$, then it is possible to write $\vec{r}$ as follows:

$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z \qquad (122)$$

Now assume some coordinate system X'Y'Z' which has the same origin as the XYZ system but an arbitrary rotation with respect to it. The components of $\vec{r}$ in this system are x', y' and z' and the unit vectors along the three axes are $\vec{i}\,'$, $\vec{j}\,'$ and $\vec{k}\,'$. The vector $\vec{r}$ may also be written

[17] [OW: original only had "i," here]

$$\vec{r} = \vec{i}\,'x + \vec{j}\,'y + \vec{k}\,'z \ . \tag{123}$$

The problem is to determine the components x', y', and z' in terms of x, y and z and the relative orientations of the two coordinate systems. This process is called an orthogonal transformation. It is possible to write the unit vector $\vec{i}\,'$ in terms of its components in the XYZ system,

$$\vec{i}\,' = \left(\vec{i}\,'\cdot\vec{i}\right)\vec{i} + \left(\vec{i}\,'\cdot\vec{j}\right)\vec{j} + \left(\vec{i}\,'\cdot\vec{k}\right)\vec{k} \tag{124}$$

Since all these vectors have unit magnitude, the dot product of two is simply the cosine of the angle between them.

$$\begin{aligned} \vec{i}\,'\cdot\vec{i} &= \cos\angle\,\vec{i}\,'\cdot\vec{i} = a_{11} \ , \\ \vec{i}\,'\cdot\vec{j} &= \cos\angle\,\vec{i}\,'\cdot\vec{j} = a_{12} \ , \\ \vec{i}\,'\cdot\vec{k} &= \cos\angle\,\vec{i}\,'\cdot\vec{k} = a_{13} \ . \end{aligned} \tag{125}$$

The same process may be applied in obtaining $\vec{j}\,'$ and $\vec{k}\,'$.

$$\vec{j}\,' = \left(\vec{j}\,'\cdot\vec{i}\right)\vec{i} + \left(\vec{j}\,'\cdot\vec{j}\right)\vec{j} + \left(\vec{j}\,'\cdot\vec{k}\right)\vec{k}$$

$$\vec{k}\,' = \left(\vec{k}\,'\cdot\vec{i}\right)\vec{i} + \left(\vec{k}\,'\cdot\vec{j}\right)\vec{j} + \left(\vec{k}\,'\cdot\vec{k}\right)\vec{k}$$

so the entire set of relationships may be written:

$$\begin{aligned} \vec{i}\,' &= a_{11}\vec{i} + a_{12}\vec{j} + a_{13}\vec{k} \ , \\ \vec{j}\,' &= a_{21}\vec{i} + a_{22}\vec{j} + a_{23}\vec{k} \ , \\ \vec{k}\,' &= a_{31}\vec{i} + a_{32}\vec{j} + a_{33}\vec{k} \ . \end{aligned} \tag{126}$$

It is possible to apply an exactly similar process in expressing the unit vectors $\vec{i}\,'$, $\vec{j}\,'$ and $\vec{k}\,'$ in terms of their components in the X'Y'Z' system.

$$\begin{aligned} \vec{i} &= a_{11}\vec{i}\,' + a_{21}\vec{j}\,' + a_{31}\vec{k}\,' \ , \\ \vec{j} &= a_{12}\vec{i}\,' + a_{22}\vec{j}\,' + a_{32}\vec{k}\,' \ , \\ \vec{k} &= a_{13}\vec{i}\,' + a_{23}\vec{j}\,' + a_{33}\vec{k}\,' \ . \end{aligned} \tag{127}$$

Figure 9 shows the two coordinate systems and the unit vectors.

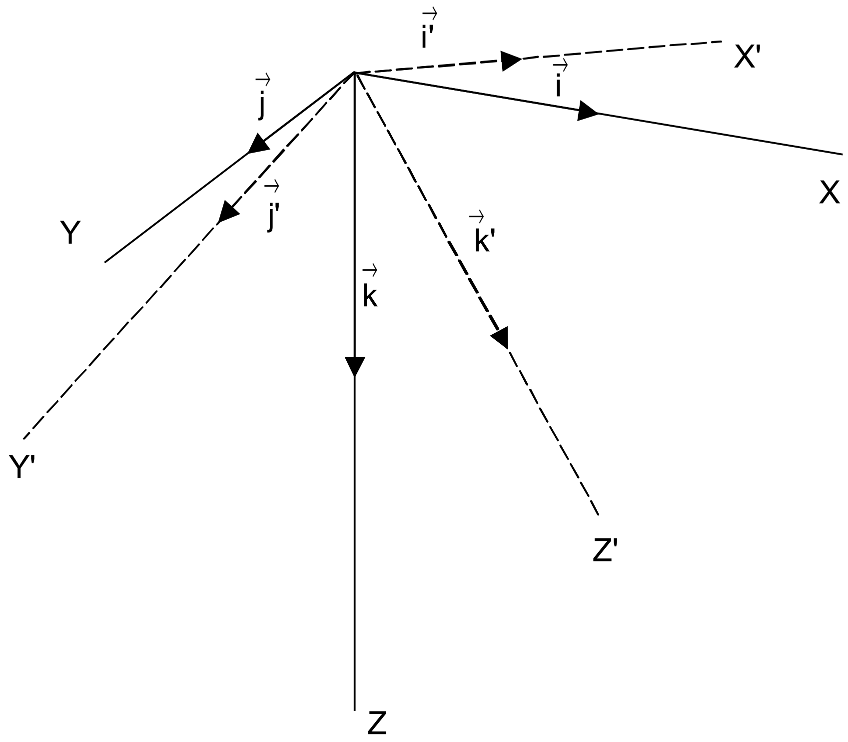

Figure 9

It is now possible to determine the components of the vector $\vec{r}$ in the X'Y'Z' coordinate system.

$$
\begin{aligned}
x' &= \vec{r}\cdot\vec{i}' = a_{11}x + a_{12}y + a_{13}z \\
y' &= \vec{r}\cdot\vec{j}' = a_{21}x + a_{22}y + a_{23}z \\
z' &= \vec{r}\cdot\vec{k}' = a_{31}x + a_{32}y + a_{33}z
\end{aligned} \tag{128}
$$

The nine quantities a_{11} - - a_{33} are called the direction cosines. They provide the means of transforming a vector from one coordinate system to another and therefore they specify the orientation of the X'Y'Z' system with respect to the XYZ system. It was developed earlier that only three parameters were necessary to specify the orientation of a rigid body. Therefore there must be six equations relating the direction cosines to each other. It will be noted that regardless of what rotation is applied to the coordinate system, the length of any vector must remain unchanged. This means that

$$(x')^2 + (y')^2 + (z')^2 = x^2 + y^2 + z^2 \,. \tag{129}$$

Substitution of the Equations (128) into this equation shows that if Equation (129) is to hold identically for all values of x, y and z, then the following conditions must obtain:

$$
\begin{aligned}
a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1 \;, \\
a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1 \;, \\
a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1 \;, \\
a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0 \;, \\
a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0 \;, \\
a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0 \;.
\end{aligned} \tag{130}
$$

These six equations are called the orthogonality conditions. The entire set of equations may be written in condensed form as

$$\sum_i a_{ij} a_{jk} = \delta_{jk} \quad , \tag{131}$$

where δ_{jk} is the Kronecker δ-symbol which is defined by

$$\begin{aligned} \delta_{jk} &= 1 \ , \ (j = k) \\ \delta_{jk} &= 0 \ , \ (j \neq k) \ . \end{aligned} \tag{132}$$

It will be noted that the nine direction cosines, restrained by the six orthogonality equations, give the three independent parameters necessary to define the orientation of a rigid body. The nine direction cosines may be written in an array called a matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = (A) \ . \tag{133}$$

Matrices are a type of mathematical entity which may be conveniently applied to the problem of rigid body rotations. The rules for manipulating these quantities will now be reviewed.

3. Properties of Matrices

The multiplication of a matrix by a vector is the first operation of interest. Symbolically, this is represented by

$$\vec{r}' = (A)\vec{r} \tag{134}$$

For convenience, the x, y and z components of $\vec{r}$ are denoted by x_1, x_2, and x_3. Note that a vector $\vec{r}$ may be viewed as a matrix of only one column. The equation might be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{135}$$

The rule for performing this operation is

$$x_i' = \sum_{j=1}^{3} a_{ij} x_j \tag{136}$$

If these operations are carried out, a set of three equations is obtained which is identical with the set of Equations (128). This means that multiplication of a vector by a matrix using the multiplication rule above represents a transformation of that vector from one coordinate system to another. For this reason, the matrix (A) may be called the transformation matrix.

The case of two successive rotations is an important one. Let the first rotation be represented by a matrix (B). Then the components of a vector after this rotation will be given by

$$x_k' = \sum_j b_{kj} x_j \tag{137}$$

If the second rotation is represented by the matrix (A), then the components of the vector after this second rotation would be

$$x_i'' = \sum_k a_{ik} x_k' \tag{138}$$

Substituting (137) into (138) gives

$$x_i'' = \sum_k a_{ik} \sum_j b_{kj} x_j \ ,$$

$$= \sum_j \left(\sum_k a_{ik} b_{kj} \right) x_j \ .$$

Note that this can be put in the form of Equation 136.

$$x_i'' = \sum_j c_{ij} x_j \ , \qquad (139)$$

where

$$c_{ij} = \sum_k a_{ik} b_{kj} \ . \qquad (140)$$

Thus the two rotations may be replaced by a single rotation (C), the elements of which may be computed from (140). Symbolically,

$$(C) = (A)\,(B)\ . \qquad (141)$$

It can be seen by the rule of Equation (140) that

$$(A)\,(B) \neq (B)\,(A)\ ,$$

so the process of matrix multiplication is not commutative. The process of matrix multiplication is asociative:

$$(A)\,[(B)\,(C)\} = [(A)\,(B)]\,(C)\ .$$

The matrix (A) was used to transform the vector $\vec{r}$ into the vector $\vec{r}\,'$.
It is of interest now to investigate the properties of the matrix $(A)^{-1}$ which transforms $\vec{r}\,'$ into $\vec{r}$. The elements of this inverse matrix are designated by a_{ij}' . The inverse matrix is defined by the following equation.

$$(A)^{-1}(A)\vec{r} = \vec{r} \ . \qquad (142)$$

Doing the first operation, the result is

$$x_i' = \sum_j a_{ij} x_j \ . \qquad (143)$$

Now applying the inverse transformation to this gives

$$x_k'' = \sum_i a_{ki}' x_i' \ ,$$

$$= \sum_i a_{ki}' \sum_j a_{ij} x_j \ ,$$

$$x_k'' = \sum_j \left(\sum_i a_{ki}' a_{ij} \right) x_j \ .$$

Now according to the requirement that this must give back the original vector, $x_k'' = x_k$. This will be true only if

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This shows that the product of the two matrices (A) and (A)~ will be

This matrix (I) is called the identity matrix. It may be easily shown from the rules of matrix multiplication that for any matrix (Q),

Now since (A)" corresponds to some physical rotation, there must exist some matrix (R) which is the inverse of (A)" . In other words, there must be an (R) such that

Now if (R) is applied to both sides of Equation (145), the result is

(148)

Since matrix multiplication is associative. Equation (147) may be substituted into Equation (148) to give

This means that

so that (A) and (A)" commute. Now consider the double sum,

This sum may be written two ways, depending on the order of summation,

Applying Equation (144) to the quantity in parentheses on the left hand side, and applying the orthogonality condition of Equation (131) to the quantity in parentheses on the right hand side, the result is

This is the important result. To form the inverse of an orthogonal matrix, the rows and columns are simply interchanged. Note that this conclusion holds true only for orthogonal matrices. This is because the orthogonality conditions were used to prove Equation (152). In general, the matrix formed by interchanging rows and columns is called the transposed matrix and is desig. nated by (A). The complex conjugate of this transposed matrix is called the adjoint matrix and is indicated by (A) = (A) . A matrix is said to be unitary if it satisfies the condition,

153)

Of course these latter definitions are relatively meaningless in the case of real matrices. However, use is sometimes made of matrices, the elements of which are complex numbers.

It is of interest to investigate the characteristics of the determinant formed by the elemenCof a matrix. The determinant of the matrix (A) will be written as [A]. It will be noted that the law of matrix multiplication is the same as the law for multiplication of determinants. Therefore,

154)

Evidently the determinant of the identity matrix has the value unity, therefore, from Equation (145) it may be seen that

155)

provided that (A) is orthogonal. Since interchanging rows and columns does not alter the value of a determinant, [A~] = [A] and, from Equation (155),

156)

This means that the determinant of the transformation matrix can have only the values plus or minus one. If the rotation is a real one, it may be shown that +1 is the only allowable value. There is a certain type of matrix opera. tion which is called a similarity transformation. It is defined by

157)

It can easily be shown that the determinant of (A) is the same as the determinant of (A)', that is to say, the value of the determinant of a matrix is invariant under a similarity transformation of that matrix. This may be shown by simply applying both sides of (157) to the matrix (B).

From this it is seen that

Since [B] is a number and not zero, it is possible to divide both sides by it and obtain the result

[A]=[A'j, (160) which demonstrates the proposition.

There is another set of relationships among the direction cosines which will prove to be of interest. Consider the set of Equations (126). If the i', j' and k' vectors are mutually perpendicular, then the following relationships apply:

If these vector equations are expanded in the unprimed system, and their components equated, the result is

162)

These nine equations are really consequences of the orthogonality conditions. They present a means for solving for any direction cosine in terms of the others.

4. Infinitesmal Rotations

It would be a great advantage if a vector could be associated with a finite rotation, but it turns out that this is not possible. For one thing, finite rotations are not commutative, nor even anti-commutative. That is to say the order of the operations must be preserved. While this is true of a finite rotation, it will be shown that a vector may be associated wit'n an infinitesimal rotation and that therefore, the known characteristics of vectors may be used in the treatment of such rotations.

Consider the matrix that describes a rotation thru the angle .241,, about a line which makes the angles a,, p. and -y, with the X, Y and Z axes respectively. This matrix may be gotten by substituting into the matrix (12) and dropping higher order terms. The result is

(163)

This matrix differs only slightly from the identity matrix. This may be seen more clearly by writing it in the following form:

(164)

This latter matrix is anti-symmetric or skew-symmetric. Notice that this matrix has only three independent elements, ^*i,* cos a. ; Ap., cos p, ;

Ap.. cos -Y. and that they are simply the three components of a vector of magnitude ^p. which is oriented along the axis of rotation. It will be shown that this is the vector which may be associated with infintesmal rotation. Let these three components be called f2., 0-,, f2, so that (A^l) may be written

Now if the infinitesmal rotation (A), is followed by another infinitesmal rotation (A)' , of the form

(166)

then the combined rotation (A). (A), is seen to be the following, if higher order infinitesmals are dropped:

where

Since the second order infinitesmals were dropped, the order or sequence of the infinitesmal rotations is unimportant. This is one condition which is necessary if these rotations are to be represented by vectors. From the makeup of n" , n", , n", , it is seen that the vector representing the combined rotation is simply the sum of the two vectors for the single rotations.
A more conclusive demonstration of the fact that the quantities S7., *Q-,,* Q, are the components of a vector associated with the infinitesmal transformation is the demonstration that the matrix components transform like components of a vector under a coordinate transformation. Consider a matrix (A) which operates on a vector R to produce a vector R'.
Now if an additional matrix (B) is applied to this equation,
This equation is simply Equation (168) when seen in a different coordinate system, and (B) (A) (B)" is the matrix (A) when viewed from the different coordinate system. This is the similarity transformation, which has been introduced before. If a similarity transformation is applied to the 'matrix of Equation (l65), the result is
-1

Expanding and equating components,
Thus, the infinitesmal transformation, when viewed from the other coordinate system defined by (B) is still nearly the identity transformation, and the vector which represents the vector associated with the infinesmal transformation in this new system is simply the transform of the vector representing the infinitesmal transformation in the other coordinate system. This shows the vector character of the set of elements 0., Q,, f2,.
By using this infinitesmal transformation, the rate of change of the transformation matrix (A) may be found in much the same way that the derivative of the matrix (H) was established in Section III. If (A) is the matrix at the beginning of time interval, and (A) is the matrix at the end of time At, then the derivative of (A) is given by
(A)' may be viewed as the rotation (A) followed by the infinitesmal transformation going from (A) to (A)'. In other words
where
Again, in the limit -=^- is simply the rate of rotation, and -r^- cos a = P, -^- cos p = Q, -r^- cos -Y = R, so the equation becomes
Expanding, and equating components gives
These are the rates of change of the direction cosines in terms of the angular velocity. Now if Equation (175) be multiplied on the right by the transpose of (A), the result is
Expanding and equating components gives the following relationships:
It is interesting that two different expressions are obtained for each of the velocity components. This is a consequence of the great amount of redundancy in the direction cosines. The equivalence of the two expressions for any one of the components may be shown by making use of Equation (l62).

APPENDIX B - THE EULER ANGLES

It was demonstrated in Appendix A that three parameters were required to fix the orientation of a rigid body and hence of a coordinate system. The nin^ direction cosines do not lend themselves to a reduction to three simple parameters, nor do they give a very lively picture of the orientation of the body. Both these difficulties are overcome by use of Euler angles, the only three-par. . meter system in common use. In this method, a rotation is represented by thr«:» individual rotations taken in a specified sequence about certain specific axes. *If.* the literature, there is no agreement whatever on the order of rotations, the ax»-» about which the rotations are made, or notation. These are varied to suit the needs of the problem and/or the authors whim. Texts on classical mechanics give sets of angles defined so as to facilitate solution of the spinning top problem. The system presented here is the most common, though by no means the only one used in aircraft work.

Consider two coordinate systems initially coincident. One set of coordinates, the x, y, z, will be referred to as the fixed system, and the other will move w'lrh respect to it. The first rotation is through the angle ^ about the z' axis. This is shown in Figure 10.

The second rotation is through the angle 9 and is done about the Y' axis and the resulting axis system is called X", Y", Z". This rotation is shown in Figure 11. 9 is commonly called the pitch angle. The final rotation is done about the X" axis through the angle 4>. This is called the roll angle and all three rotations are shown in Figure 12. Note that all three of these rotations are in the positive sense. That is to say if the thumb of the right hand is placed along the axis of rotation, the direction of rotation is that direction in which the curled fingers point.

It is now necessary to determine the transformation matrix in terms of these Euler angles. It was shown earlier that successive rotations could be represented by a matrix which is a product of the matrices of the individual rotations. It is necessary then, only to compute the matrix corresponding to each of the Euler angle rotations and to multiply them together in the appropriate order. Note that each of the rotations is simply a two-dimensional transformation because in each case the rotation is about one of the moving axes and hence components along that axis are unchanged.

Consider first, the rotation through the angle which is shown in Figure 10. If this is viewed from above, the transformation of some arbitrary vector R would appear as shown in Figure 13

It can be seen from the geometry of Figure 13 that the new x' and y' components are related to the old by the equations

Since the rotation was about the z axis, any z component of R would remain unchanged. In other words, Z = Z'. This fact, together with the Equation (179) shows that the matrix for the rotation is

Now the rotation of Figure 11 may be viewed from the front along the Y" axis,

and Figure 14 is obtained.

ii

From the geometry of the above figure, it may be seen that

In this rotation, the Y components remain unchanged so that Y'' = Y''. Therefore, the matrix for this rotation is

The final rotation may be viewed from the front, looking along the X'' axis of Figure 12. From the geometry of this figure it is seen that

In this rotation, the X components remain unchanged so that X''' = X''. Therefore, the transformation matrix for this rotation is

In order to get the total transformation matrix which results from these three rotations, it is only necessary to multiply the three individual matrices in the correct order.

By comparison of this matrix with the matrix (A) it may be seen that all of the direction cosines and hence the complete transformation, can be expressed in terms of the three independent parameters ^*i,* 9, 4>.

Since the position of a coordinate system may be specified in terms of Euler angles, the rate of rotation of that coordinate system must be related to the rates of change of the Euler angles. We now investigate this relationship.

It is shown in Appendix A that a vector could be associated with a rate of rotation. This vector is along the instantaneous axis of rotation and is equal in magnitude to the rate of rotation. Thus, each of the Euler angle rates may be associated with a vector along the axis of rotation. Observe that the vector associated with the + rotation of Figure 10 is directed along the Z axis and

·

points downward if ^*i is* positive. Similarly, the rate of rotation due to the 9 rotation of Figure 11 is a vector along the Y' axis, and if 9 is increasing, the vector is in the positive y' direction. Finally, a positive roll rotation is a vector directed along the X''' axis of Figure 12. The three vectors representing the threi individual Euler angles rates must be added together in order to get the entire rate of rotation of the system. Recall that these vectors are added according to the usua vector rule. The situation is shown in Figure 16 where all the Euler angle rates ar assumed positive. Note that these three vectors are not mutually orthogonal. The vector is normal to the 9 vector, and the 9 vector is normal to the vector, but the vector is not normal to the + vector. In any case, the three may be transformed into the X''' Y''' Z''' and added to give the entire velocity vector. the 4' vector has the components 0, 0, ^ in the XYZ system, so to transform this into the X''' Y''' Z''' system, it is necessary to apply the full transformation matrix (185) to this vector. If this is done, the result is

Now the vector 9 has the components 0, 9, 0 in the X'' Y'' Z'' coordinate system.

In order to get this into the X''' Y''' Z''' system, it is only necessary to transform through the last of the Euler angle rotations which is defined by the matrix (184). If this is done, the result is

The vector 4>, of course, is already in the X''' Y''' Z''' system, being defined by

In order to get the entire velocity vector, it is only necessary to add the last three equations. If this is done, and if the total angular velocity vector is defined as "^ = "T"[1] P + "J"' Q + 'K"' R, then,

These three equations may be solved for ^*i,* 9, 4> giving

From these equations, it is easier to see the difficulties which arises^, when 9 approaches 90 . For this value of 9, both 4' s^d 4> are infinite. It is interesting to note t'hat the derivative of 9 itself has no such anomalies.